Boundedness of relative complements & anti-canonical volume.

Theorem 1.7 Let d be a natural number and $\mathcal{R} \subset [0,1]$ be a finite set of rational numbers. Then, there exists a natural number n depending only on d and $\mathcal{R}$ satisfying the following

Assume (X,B) is a projective pair such that:

- (X,B) is lc of dimension d.
- $\text{coeff}(B) \subset \Phi(\mathcal{R}) = \{1 - \frac{r}{n} \mid r \in \mathcal{R}, n \in \mathbb{N}\}$
- X is of Fano type, and
- $-(K_X+B)$ is nef.

Then, there is an n-complement $\boxed{K_X+B^+}$ of K_X+B such that $B^+ \geq B$. Moreover, the complement is an mn-complement for any $m \in \mathbb{N}$.

Theorem 1.8 Let d be a natural number and $\mathcal{R} \subset [0,1]$ be a finite set of rational numbers. Then, there exists a natural number n depending only on d and $\mathcal{R}$ satisfying the following.

Assume (X,B) is a pair and $X \to Z$ is a contraction such that:

- (X,B) is lc of dimension d and $\dim Z > 0$.
- $\text{coeff}(B) \subset \Phi(\mathcal{R})$
- X is of Fano type over Z, and
- $-(K_X+B)$ is nef over Z.

Then, for any point $z \in Z$, there is an n-complement K_X+B^+ of K_X+B over z such that $B^+ \geq B$. Moreover it is also an mn-complement for any $m \in \mathbb{N}$.

9 Boundedness of relative complements.

Proposition 8.1 Assume Theorem 1.7 and 1.8 hold in dimension $d-1$. Then, Theorem 1.8 holds in dimension d for those (X,B) and $X \to Z$ s.t.

- $\operatorname{coeff}(B) \subset \mathcal{R}$,
- (X,Γ) is $\mathbb{Q}$-factorial plt for some Γ,
- $-(K_X+\Gamma)$ is ample over Z
- $S := \lfloor \Gamma \rfloor$ is irreducible and it is a component of $\lfloor B \rfloor$, and
- S intersects the fiber of $X \to Z$ over z.

Proof $f: X \to Z$.

Step 1 Show that $S \to f(S)$ is a contraction.

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0.$$

$$f_*\mathcal{O}_X \to f_*\mathcal{O}_S \to R^1f_*\mathcal{O}_X(-S) = 0$$

$$-S = \underbrace{K_X+\Gamma-S}_{\substack{(X,\Gamma-S)\\ \text{klt.}}} \underbrace{-(K_X+\Gamma)}_{\text{ample.}}$$

$$f_*\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_S.$$

Stein factorization $S \to V \xrightarrow{\pi} Z$.

$$\underbrace{\mathcal{O}_Z}_{} = f_* \mathcal{O}_X \twoheadrightarrow f_* \mathcal{O}_S = \pi_* \mathcal{O}_V.$$

$\mathcal{O}_Z \twoheadrightarrow \pi_* \mathcal{O}_V$. factors as $\mathcal{O}_Z \to \underbrace{\mathcal{O}_{f(S)} \to \pi_* \mathcal{O}_V}$.

is surjective.

isomorphism $\mathcal{O}_{f(S)} \to \pi_*(\mathcal{O}_V) = f_* \mathcal{O}_S$.

S, $S \to f(S)$ is a contraction.

Step 2) log res. $\phi: X' \to X$ of (X, B)

$\psi: S' \to S$ the bir. transform.

Adjunction. $K_S + B_S := (K_X + B)|_S$

coeff $B_S \subset \underset{\text{finite depending on } \mathcal{R} \text{ and } d.}{\Phi(\mathbb{S})}$:

1.7 or 1.8) can be applied $S \to \underline{f(S)}$.

$\exists\ \underbrace{B_S^+}$, $\underbrace{n}$-complement of B_S.

we increase n s.t $\underbrace{nB}$ is integral.

Step 3) Notation: $(K_{X'} + B') := \phi^*(K_X + B)$

$(K_{X'} + \Gamma') := \phi^*(K_X + \Gamma)$.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

$T' := \lfloor B'^{\geq 0} \rfloor \quad \Delta' := B' - T' = B'^{<1}.$

$L' := -nK_{X'} - nT' - \lfloor (n+1)\Delta' \rfloor$ } L' is integral.

$= n\Delta' - \lfloor (n+1)\Delta' \rfloor - n(K_{X'} + B')$

* [replace Γ with $(1-a)\Gamma + aB$ for a close to. 1 $\Rightarrow$ $B' - \Gamma'$ has small coeff.]

Step 4) P'.

$\mu_{S'} P' = 0$, μ_S = coeff at S'

for $D' \neq S'$.

$$\mu_{D'} P' = -\mu_D(\lfloor \Gamma' + n\Delta - \lfloor (n+1)\Delta' \rfloor \rfloor)$$

equivalent to. P' the unique integral div. s.t.

$$(X', \underbrace{P' + \Gamma' + n\Delta' - \lfloor (n+1)\Delta' \rfloor}_{\Lambda}) \text{ is plt.} \qquad \lfloor \Lambda \rfloor = S'.$$

1st) $0 \leq \mu_{D'} P' \leq 1$.

2nd) P' is exceptional $/X$.

Proof) D' not exc. $/X$. nB integral.

$\Rightarrow \mu_{D'}(n\Delta') \in \mathbb{Z}$.

$$\mu_{D'}(n\Delta' - \lfloor (n+1)\Delta' \rfloor) = 0 \Rightarrow \mu_{D'}(P') = -\mu_{D'} \lfloor \Gamma' \rfloor$$

$\mu_{S'}(P') = 0$ by def. $\ddot{0}$

Step 5] We can lift sections. S' to X'.

$$L' + P' = \underbrace{n\Delta' - \lfloor (n+1)\Delta' \rfloor} - n(K_X + B') + \underbrace{P'} \quad + \Gamma' = \Lambda'$$

$$= \underbrace{K_{X'} + \Lambda'} - \underbrace{\underbrace{(K_{X'} + \Gamma')}_{\phi^*(\text{ample})/Z} - \underbrace{n(K_{X'} + B')}_{\text{nef}/Z}}_{\text{big and nef}}$$

$(X', \Lambda' - S')$ is klt

$\Rightarrow h^1(L' + P' - S') = 0$ rel. K-V.

$$H^0(L' + P') \twoheadrightarrow H^0(L' + P'|_{S'})$$

Step 6 $R_S := B_S^+ - B_S$ effective.

$$-n(K_S+B_S) = -n(\underbrace{K_S+B_S^+}_{0} + \underbrace{B_S - B_S^+}) \sim nR_S \geq 0.$$

$$\underline{(L'+P')|_{S'}} = (\underbrace{n\Delta' - \lfloor (n+1)\Delta' \rfloor - n(K_{X'}+B')}_{L' \text{ by def.}} + P')|_{S'}$$

$$\sim \underbrace{nR_{S'} + n\Delta_{S'} - \lfloor (n+1)\Delta_{S'} \rfloor + P_{S'}}_{G_{S'}}.$$

Step 7 $G_{S'}$ is effective. so we can lift it to

$G' \geq 0$ on X'. $G' \sim L'+P'$.

Step 8 $0 \leq G \sim L + \underset{\text{exc.}/X.}{P} = L = -nK_X - nT - \lfloor (n+1)\Delta \rfloor$

$$\begin{aligned} \lfloor (n+1)\Delta \rfloor &= n\Delta \\ -nK_X - \underbrace{nT - n\Delta} \\ &= -nK_X - nB \\ &= -n(K_X+B) \end{aligned}$$

$\boxed{nB \text{ is integral}} \Rightarrow \lfloor (n+1)\Delta \rfloor = n\Delta$

$nR := G \Rightarrow -n(K_X+B+R) \sim 0.$

$B^+ := B+R.$ will be the complement.

Step 9) (X, B^+) is lc over Z.

$$K_S + B_S^+ = (K_X + B^+)|_S$$ by inv. of adj.

(X, B^+) is lc near S.

$\Omega := aB^+ + (1-a)\Gamma$ for a close to 1.

If (X, B^+) is not lc over Z.

$\Rightarrow (X, \Omega)$ is not lc either.

$$-(K_X + \Omega) = \underbrace{-\underbrace{a(K_X + B^+)}_{\text{nef}} - \underbrace{(1-a)(K_X + \Gamma)}_{\text{ample}}}_{\text{ample}}.$$

non-klt locus is connected for (X, Ω).

S and non-lc center have to be disconnected

contradicts the connectedness principle.

$\square$

Proposition 8.2 If Theorem 1.7 & 1.8 hold in dimension $d-1$.
Then Theorem 1.8 holds in dimension d.

Proof $f: X \to Z$.

Step 1 Reduce to $\lfloor B \rfloor$ has a vertical component intersecting the fibre over z.

- Cartier N on Z passing through z.

$t := \operatorname{lct}(f^*N, (X, B))$ over z.

$\Omega := B + t f^*N$

(X, Ω) lc. (X', Ω') dlt model of (X, Ω)

$\Rightarrow X'$ of Fano type over Z.

$\exists \Delta' \leq \Omega'$ s.t. $\Delta' \in \Phi(\mathcal{R})$.
with a vertical component.

$-(K_{X'} + \Delta')$-MMP over Z. $\to X''$.

$$-(K_{X'} + \Delta') = \underset{\text{nef}/Z}{-(K_{X'} + \Omega')} + \underset{\geq 0}{(\Omega' - \Delta')}$$

MMP ends with $\underbrace{-(K_{X''} + \Delta'') \text{ nef}/Z.}$

(X'', Δ'') is lc. we cannot contract components of $\lfloor \Delta' \rfloor$ by this MMP.

Replace (X, B) for (X'', Δ'').

Step 2) Reduce to $\operatorname{coeff}(B) \in \mathcal{R}$.

We can replace the coeff of $B \subset (1-\varepsilon, 1)$ by 1. (we have finite sets).

and we can use this up to running an MMP.

Step 3) Define Δ

for D vertical $\boxed{\mu_D(\Delta) := \mu_D(B)}$

for D hor. $\mu_D(\Delta) := \alpha \mu_D(B)$ $\alpha < 1$ close to 1.

$\alpha B \leq \Delta \leq B$. $\Rightarrow (X, \Delta)$ is l.c.

$-(K_X + \Delta)$ is big over Z.

$$\boxed{-(K_X + \alpha B) = \underbrace{\underbrace{-\alpha(K_X + B)}_{\text{nef}/Z} - \underbrace{(1-\alpha) K_X}_{\text{Big}/Z}}_{\text{big}/Z}}$$

Step 4)

$X \to V$ contraction by $(-K_X+B)$

$-(K_X+\Delta)$-MMP over V

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr] & X' \arrow[d] \\ & V \end{tikzcd}$$

we can get that

$-(K_{X'}+\Delta')$ is big and nef $/_Z$.

$$\underbrace{-(K_{X'}+B')}_{\text{big}/V} t \underbrace{-(K_{X'}+\Delta')}_{\text{ps-eff}/V}(1-t)$$

big $/V$. Δ for a different Δ.

$-(K_X+\Delta)$ big and nef.

$\bar{\Delta} := \beta\Delta$. $\beta < 1$. $(X, \bar{\Delta})$ is klt

$X \dashrightarrow T$ by $-(K_X+\Delta)$

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr] & X' \arrow[d] \\ & T \end{tikzcd}$$

$-(K_X+\bar{\Delta})$-MMP

by changing B $-(K_{X'}+\bar{\Delta}')$ is big and nef.

Step 5)

$$-(K_X+\Delta)\sim_{\mathbb{R}} \underbrace{A}_{\text{ample}}+\underbrace{E}_{\text{effective}} \quad /Z.$$

- If E contains no lc center of (X,Δ), $(X,\Delta+\delta E)$ is dlt for $\delta>0$ small.

$$-(\underbrace{K_X+\Delta}+\delta E)\sim_{\mathbb{R}} A+E-\delta E$$

$$=\underbrace{\underbrace{\delta A}_{\text{ample}}+\underbrace{(1-\delta)A+(1-\delta)E}_{\substack{(1-\delta)(A+E)\\ \| \\ \text{big and nef.}}}}_{\underline{\text{ample}}.}$$

Perturb coeff. of $\Delta+\delta E$ to obtain the desired Γ.

Otherwise we "add" δE to $\tilde{\Delta}$.

to reduce to $\underline{\underline{8.1}}$ i.e finding Γ

$\square$

§9 Anti-canonical volumes.

Theorem 1.6 Let d be a natural number $\varepsilon \in \mathbb{R}_{>0}$. Assume BAB for $d-1$

$\left\{ \begin{array}{l} \varepsilon\text{-lc } (d-1)\text{-dimensional Fanos form a bounded} \\ \text{family} \end{array} \right\}$

Then, $\exists v$ depending on d, ε, such that

X ε-lc d-dimensional weak Fano.

$$\mathrm{vol}(-K_X) \leq v.$$

Proof

Lemma 2.22 $\mathcal{P}$ a $\boxed{\text{bounded set of couples}}$ and $e \in \mathbb{R}^{>0}$. Then, $\exists I$ finite $I(\mathcal{P}, e)$ s.t.

$(X, B) \in \mathcal{P}$, $D \geq 0$ integral div.

$\boxed{K_X + B + sD \equiv 0}$ with $s \geq e$.

Then $s \in I$.

Step 1) Assume X_i, ε-lc Fano

$\mathrm{vol}(-K_{X_i}) \nearrow \infty$.

Fix $0 < \varepsilon' < \varepsilon$. $a_i \searrow 0$ rational.

$$B_i \sim_{\mathbb{Q}} -a_i K_{X_i}, \quad \boxed{(2d)^d < \mathrm{vol}(B_i)}$$

s.t. (X_i, B_i) is exactly ε'-lc.

$\exists\, D_i'$ s.t. $a(D_i', (X_i, B_i)) = \varepsilon'$.

$\phi_i : X_i' \to X_i$ contraction extracting only D_i'.

$\phi_i^* K_{X_i} =: K_{X_i'} + e_i D_i'$ $\quad e_i \leq 1 - \varepsilon$.

$\phi^*(K_{X_i} + B_i) =: K_{X_i'} + B_i'$

$\mu_{D_i'} B_i' = 1 - \varepsilon'$

$\Rightarrow \phi_i^* B_i = B_i' - e_i D_i'$ has D_i' coeff $\boxed{\geq \varepsilon - \varepsilon'}$.

Step) Get a MFS and the $s_i \nearrow \infty$ ($\Rightarrow\Leftarrow$)

- H_i general ample $K_{X_i} + B_i + H_i \sim_{\mathbb{Q}} 0$.

$$\begin{array}{ccc} X_i' & \dashrightarrow & X_i'' \\ & & \downarrow \\ & & Z_i \end{array}$$

$-D_i'$-MMP.

MFS.

$b_i := \frac{1}{a_i} - 1 \nearrow \infty$

$$K_{X_i''} + B_i'' + \underbrace{b_i \phi_i^* B_i''}_{\geq b_i(\varepsilon - \varepsilon'')} \sim_{\mathbb{Q}} K_{X_i''} + B_i'' + b_i B_i''$$

$$\sim K_{X_i''} + B_i'' + H_i'' \sim_{\mathbb{Q}} 0.$$

D_i'' coeff.

$\Rightarrow s \geq b_i(\varepsilon - \varepsilon'')$

s.t. $\boxed{K_{X_i''} + s D_i'' \sim_{\mathbb{Q}} 0 / Z_i.}$

If $\dim Z_i > 0$.

In the fibers we can apply BAB and get a contradiction with 2.22.1. $\square$